The Aronsson equation for absolute minimizers of L^∞-functionals associated with vector fields satisfying Hörmander's condition

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Abstract. *Given a Carnot-Carathéodory metric space (R^n, d_{CC}) generated by vector fields $\{X_i\}_{i=1}^m$ satisfying Hörmander's condition, we prove in theorem A that any absolute minimizer $u \in W^{1,\infty}_{CC}(\Omega)$ to $F(v,\Omega) = \sup_{x\in\Omega} f(x, Xv(x))$ is a viscosity solution to the Aronsson equation (1.6), under suitable conditions on f. In particular, any AMLE is a viscosity solution to the subelliptic ∞-Laplacian equation (1.7). If the Carnot-Carathédory space is a Carnot group $\mathbf{G}$ and f is independent of x-variable, we establish in theorem C the uniquness of viscosity solutions to the Aronsson equation (1.13) under suitable conditions on f. As a consequence, the uniqueness of both AMLE and viscosity solutions to the subelliptic ∞-Laplacian equation is established in $\mathbf{G}$.*

§1. Introduction

Variational problems in L^∞ are very important because of both its analytic difficulties and their frequent appearance in applications, see the survey article [B] by Barron . The study began with Aronsson's papers [A1, 2]. The simplest model is to consider minimal Lipschitz extensions (or MLE): for a bounded, Lipschitz domain $\Omega \subset R^n$ and $g \in \mathrm{Lip}(\Omega)$, find $u \in W^{1,\infty}(\Omega)$, with $u|_{\partial\Omega} = g$, such that

$$\|Du\|_{L^\infty(\Omega)} \le \|Dw\|_{L^\infty(\Omega)}, \ \forall w \in W_0^{1,\infty}(\Omega), \text{ with } w|_{\partial\Omega} = g. \tag{1.1}$$

Since MLE's may be neither unique nor smooth, Aronsson [A1] introduced the notation of absolutely minimizing Lipschitz extensions(or AMLE for short), and proved that any C^2 AMLE solves the ∞-Laplacian equation

$$\Delta_\infty u := -\sum_{ij=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \quad \text{in } \ \Omega. \tag{1.2}$$

However, (1.2) is a highly nonlinear and highly degenerate PDE and may not have C^2 solutions in general. This issue was finally settled by Jensen [J2], who not only established the equivalence between the AMLE property and the solution to eqn.(1.2) in the viscosity sense, which was first introduced by Crandall-Lions [CL](see also Crandall-Ishii-Lions [CIL]), but also proved the uniqueness of viscosity solutions to eqn.(1.2) with the

Dirichlet boundary value. The remarkable analysis of [J2] involves approximation by p-Laplacian and Jensen's earlier work [J1] on the maximum principle for semiconvex functions. The reader can consult with Evans [E] and Lindqvist-Manfredi [LM] for qualitative estimates on ∞-harmonic functions. Crandall-Evans-Gariepy [CEG] developped the comparison principle of the eqn.(1.2) with *cones*, which are solutions of the eiknonal equation of forms $a + b|x - x_0|$, and gave an alternative, direct proof of the equivalence between AMLE and viscosity solution to eqn.(1.2). Furthermore, Crandall-Evans [CE] has utilized this property in their study on the regualarity issue of ∞-harmonic functions. Recently, Barron-Jensen-Wang [BJW] considered general L^∞-functionals

$$F(u, \Omega) := \sup_{x\in\Omega} f(x, u(x), Du(x)), \ \forall u \in W^{1,\infty}(\Omega).$$

and proved, under suitable conditions, that any absolute minimizer of $F(\cdot, \Omega)$ is a viscosity solution to the Aronsson-Euler equation

$$-\sum_{i=1}^{n} f_{p_i}(x, u(x), Du(x)) \frac{\partial}{\partial x_i}(f(x, u(x), Du(x))) = 0, \quad \text{in} \quad \Omega. \tag{1.3}$$

Shortly after [BJW], Crandall [C] was able to give an elegant proof of an improved version of [BJW]. Through [BJW] [C], it becomes more clear that the classical solution to the Hamilton-Jacobi equation $f(x, \phi(x), D\phi(x)) - c = 0$ plays important roles in this analysis.

Since the notion of AMLE can easily be formulated in any metric space, it is a very natural and interesting problem to study AMLE in spaces with Carnot-Carathéodory metrics, which include Riemannian manifolds and Subriemannian manifolds (e.g., Heisenberg groups, Carnot groups, or more generally Hörmander vector fields, etc). There have been several works done in this direction. For example, Juutinen [J] extended the main theorems of [J2] into Riemannian manifolds. Bieske [B1,2] was able to prove that, on the Heisenberg group $\mathbf{H}^n$ or a Grushin type space, an AMLE is equivalent to a viscosity solution to the subelliptic ∞-Laplacian equation, and the uniqueness of both AMLE and viscosity solution to the subelliptic ∞-Laplacian equation. Inspired by Crandall's argument [C], Bieske-Capogna, in a recent preprint [BC], proved that any AMLE is a viscosity solution to the subelliptic ∞-Laplacian equation for any Carnot group, where the conclusion was also proved for any AMLE, which is horizontally C^1, corresponding to those Carnot-Carathédory metrics associated to *free* systems of Hörmander's vector fields.

In this paper, we are mainly interested in the derivation of Euler equation of AMLE and its uniqueness issue for any Carnot-Carathédory metric space generated by vector fields satisfying Hörmander's condition. In this direction, we are able to prove that any AMLE is a viscosity solution to the subelliptic ∞-Laplacian equation. Moreover, if the vector fields are horizontal vector fields associated with a Carnot group, then we establish

the uniqueness for both AMLE and viscosity solution to the Euler equation. In fact, these conclusions are consequences of general theorems A and C below.

In order to state our results, we first recall some preliminary facts.

Definition 1.1. For a bounded domain $\Omega \subset R^n$ and $m \geq 1$, $\{X_i\}_{i=1}^m \subset C^2(\Omega, R^n)$ are vector fields satisfying Hörmander's condition, if there is a step $r \geq 1$ such that, at any $x \in \Omega$, $\{X_i\}_{i=1}^m$ and all their commutators up to at most order r generate R^n.

Now we recall from [NSW] the Carnot-Carathédory distance, denoted as $d_{\rm CC}$, generated by $\{X_i\}_{i=1}^m$ on Ω: for any $p, q \in \Omega$,

$$d_{\rm CC}(p,q) = \inf_{A(\delta)} \delta, \tag{1.4}$$

where

$$A(\delta) := \{r : [0,\delta] \to \Omega \mid r(0) = p,\ r(\delta) = q, r'(t) = \sum_{i=1}^m a_i(t) X_i(r(t)) \text{ with } \sum_{i=1}^m a_i^2(t) \leq 1\}.$$

Moreover, $d_{\rm CC}$ satisfies: for each compact set $K \subset\subset \Omega$,

$$C_K^{-1}\|x-y\| \leq d_{cc}(x,y) \leq C_K \|x-y\|^{\frac{1}{r}},\ \forall x, y \in K,$$

where $\|\cdot\|$ is the Euclidean norm on R^n. For $u : \Omega \to R$, denote $Xu := (X_1 u, \cdots, X_m u)$ as the horizontal gradient of u. For $1 \leq p \leq \infty$, the horizontal Sobolev space is defined by

$$W^{1,p}_{\rm CC}(\Omega) := \{u : \Omega \to R \mid\ \ \|u\|_{W^{1,p}_{\rm CC}(\Omega)} \equiv \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega)} < \infty\}.$$

The Lipschitz space, with respect to the metric $d_{\rm CC}$, is defined by

$${\rm Lip}^{\rm cc}(\Omega) := \{u : \Omega \to R \mid \|u\|_{{\rm Lip}^{\rm cc}(\Omega)} \equiv \sup_{x,y\in\Omega, x\neq y} \frac{|u(x)-u(y)|}{d_{\rm CC}(x,y)} < \infty\}.$$

It was proved by [GN] (see also [FSS]) that $u \in {\rm Lip}^{\rm CC}(\Omega)$ iff $u \in W^{1,\infty}_{\rm CC}(\Omega)$.

Now we recall the definition of absolute minimizers of L^∞-functionals over $W^{1,\infty}_{\rm CC}(\Omega)$.

Definition 1.2. For any integrand function $f : \Omega \times R^m \to R_+$, let

$$F(v, \Omega) = \sup_{x\in\Omega} f(x, Xu(x)),\ \forall v \in W^{1,\infty}_{\rm CC}(\Omega).$$

A function $u \in W^{1,\infty}_{\rm CC}(\Omega)$ is an *absolute minimizer* of $F(\cdot, \Omega)$, if for any open subset $\tilde\Omega \subset \Omega$ and $w \in W^{1,\infty}_{\rm CC}(\tilde\Omega)$, with $w = u$ on $\partial\tilde\Omega$, we have

$$F(u, \tilde\Omega) \leq F(w, \tilde\Omega). \tag{1.5}$$

u is called an *absolutely minimizing Lipschitz extension* (or AMLE), with respect to the Carnot-Carathédory metric d_{CC}, if u is an absolute minimizer of $F(\cdot, \Omega)$, with $f(x,p) = \sum_{i=1}^m p_i^2$ for $(x,p) \in \Omega \times R^m$.

Formal calculations yield that an absolute minimizer $u \in W^{1,\infty}_{\mathrm{CC}}(\Omega)$ to $F(\cdot, \Omega)$ satisfies the subelliptic Aronsson-Euler equation

$$-\sum_{i=1}^m X_i(f(x, Xu(x)))f_{p_i}(x, Xu(x)) = 0, \quad \text{in} \quad \Omega. \tag{1.6}$$

In particular, the Aronsson-Euler equation of an AMLE is the subelliptic ∞-Laplacian equation

$$\Delta_\infty^{(X)} u := -\sum_{i,j=1}^m X_i u X_j u X_i X_j u = 0, \quad \text{in} \quad \Omega. \tag{1.7}$$

In order to interpret an absolute minimizer (or AMLE respectively) as a solution to the eqn. (1.6) (or (1.7) respectively), we recall the concept of viscosity solutions by Crandall-Lions [CL] (see also [CIL]) of second order degenerate subelliptic PDEs.

Let $\mathcal{S}^m$ denote the set of symmetric $m \times m$ matrices, equipped with the usual order. A function $A \in C(R^n \times R^m \times \mathcal{S}^m)$ is called degenerate subelliptic, if, for any $(x,p) \in R^n \times R^m$

$$A(x, p, M) \leq A(x, s, N), \ \forall M, N \in \mathcal{S}^m, \quad \text{with } N \leq M. \tag{1.8}$$

Let $(D^2 u)^* \in \mathcal{S}^m$ denote the *horizontal* hessian of u, defined by

$$(D^2 u)^*_{ij} = \frac{1}{2}(X_i X_j + X_j X_i)u, \ \forall 1 \leq i, \ j \leq m.$$

Now we have

Definition 1.3. For a degenerate subelliptic equation

$$A(x, Xu(x), (D^2 u)^*(x)) = 0, \quad \text{in} \quad \Omega. \tag{1.9}$$

A function $u \in C(\Omega)$ is called a viscosity subsolution to eqn.(1.9), if for any pair $(x_0, \phi) \in \Omega \times C^2(\Omega)$ such that x_0 is a local maximum point of $(u - \phi)$ then we have

$$A(x_0, X\phi(x_0), (D^2\phi)^*(x_0)) \leq 0. \tag{1.10}$$

A function $u \in C(\Omega)$ is called a viscosity supersolution to eqn.(1.9) if $-u$ is a viscosity subsolution to eqn.(1.9). Finally, a function $u \in C(\Omega)$ is a viscosity solution to eqn.(1.9) if it is both a viscosity subsolution and a viscosity supersolution to eqn.(1.9).

It is easy to check that both eqn. (1.6) and (1.7) are degenerate subelliptic. Now we are ready to state our first theorem.

Theorem A. *Suppose that* $u \in W^{1,\infty}_{CC}(\Omega)$ *is an absolute minimizer of*

$$F(v, \Omega) = \sup_{x \in \Omega} f(x, Xv(x)),$$

where $f \in C^2(\Omega \times R^m, R_+)$ *satisfies*

(f1) f *is quasiconvex in its second variable, i.e. for any* $x \in \Omega$,

$$f(x, tp_1 + (1-t)p_2) \leq \max\{f(x, p_1), f(x, p_2)\}, \ \forall p_1 \ p_2 \in R^m, \ 0 \leq t \leq 1. \tag{1.11}$$

(f2) f *is homogeneous of degree* $\alpha \geq 1$ *and* $f_p(0,0) = 0$.

Then u *is a viscosity solution to the Aronsson-Euler equation*

$$-\sum_{i=1}^{m} X_i(f(x, Xu(x))) f_{p_i}(x, Xu(x)) = 0, \quad in \ \ \Omega. \tag{1.12}$$

The ideas to prove theorem A are based on: (1) the observation of rewrite eqn.(1.12) into an euclidean form, where we can adopt Crandall's construction [C] of solutions to the Hamilton-Jacobi equation as test functions (see also [BJW]); (2) the comparison principle of Hamilton-Jacobi equations without u-dependence (see [CIL] or [BJW]).

Since $f(x,p) = \sum_{i=1}^{p} p_i^2 (\geq 0) \in C^2(\Omega \times R^m)$ satisfies both (f1) and (f2). We have, as a consequence of theorem A,

Corollary B. *Suppose that* $u \in W^{1,\infty}_{CC}(\Omega)$ *is an AMLE, with respect to the Carnot-Carathédory metric* d_{CC}. *Then* u *is a viscosity solution to the subelliptic* ∞*-Laplacian equation (1.7).*

Now we turn to the discussion on the uniqueness problem of absolute minimizers of $F(\cdot, \Omega)$ or viscosity solutions to eqn.(1.6). Although the uniqueness might be true for general vector fields satisfying Hörmander's condition, we restrict our attention to the case where the vector fields generating the Carnot-Carathédory metrics are horizontal vector fields associated with a Carnot group $\mathbf{G}$.

To describe the uniqueness results, we recall that a Carnot group of step $r \geq 1$ is a simply connected Lie group $\mathbf{G}$ whose Lie algebra g admits a vector space decomposition in r layers $g = V^1 + V^2 + \cdots + V^r$ having two properties: (i) g is stratified, i.e., $[V^1, V^j] = V^{j+1}, j = 1, \cdots, r-1$; (ii) g is r-nilpotent, i.e. $[V^j, V^r] = 0, j = 1, \cdots, r$. V^1 is called the *horizontal* layer and $V^j, j = 2, \cdots, r$, are *vertical* layers. It is well-known (cf. Folland-Stein [FS]) that the exponential map, $\exp : g \to \mathbf{G}$, is a global differmorphism so that we

can identify $\mathbf{G}$ with $g \equiv R^n$ via exp. and $\mathbf{G}$ has an exponential coordinate system, here $n = \dim(\mathbf{G})$ is the dimension of $\mathbf{G}$. More precisely, Let $X_{i,j}$ for $1 \le i \le m_j = \dim(V^j)$ be a basis of V^j for $1 \le j \le r$, which is orthonormal with respect to an arbitrarily chosen Euclidean norm $\|\cdot\|$ on g, with respect to which the V^j's are mutually orthogonal. Then $p \in \mathbf{G}$ has coordinate $(p_{ij})_{1\le i\le m_j, 1\le j\le r}$ if $p = \exp(\sum_{j=1}^r \sum_{i=1}^{m_j}(p_{ij}X_{i,j})$. Let $\cdot$ denote the group multiplication on $\mathbf{G}$. Then it is known ([FS]) that the group law $(x,y) \to x\cdot y$ is a polynomial map with respect to the exponential map. From now on, we set $m = m_1 = \dim(V^1)$ and denote $X_i = X_{i,1}$ for $1 \le i \le m$. Two bi-Lipschitz equivalent metrics, on $\mathbf{G}$, we need are: (1) the Carnot-Carathédory metric d_{CC} on $\mathbf{G}$ generated by $\{X_i\}_{i=1}^m$; (2) the gauge metric d on $\mathbf{G}$ given as follows. For $p = (p_{ij})_{1\le i\le m_j, 1\le j\le r}$,

$$\|p\|_{\mathbf{G}}^{2r!} = \sum_{j=1}^{r}(\sum_{i=1}^{m_j}|p_{ij}|^2)^{\frac{r!}{j}},$$

with the induced gauge distance

$$d(x,y) = \|x^{-1}y\|_{\mathbf{G}}, \quad \forall x, y \in \mathbf{G}.$$

satisfying the invariant property

$$d(z\cdot x, z\cdot y) = d(x,y), \quad \forall x, y, z \in G.$$

Now we mention the Heisenberg group $\mathbf{H}^n$, which is the simplest Carnot group of step two. $\mathbf{H}^n \equiv \mathbf{C}^n \times R$ endowed with the group law: for $(z_1, \cdots, z_n, t), (z_1', \cdots, z_n', t') \in \mathbf{C}^n \times R$

$$(z_1, \cdots, z_n, t)\cdot(z_1', \cdots, z_n', t') = (z_1 + z_1', \cdots, z_n + z_n', t + t' + 2\mathrm{Im}(\sum_{i=1}^{n} z_i \bar{z_i'})),$$

whose Lie algebra $h = V_1 + V_2$ with $V_1 = \mathrm{span}\{X_i, Y_i\}_{1\le i\le n}$ and $V_2 = \mathrm{span}\{T\}$, where

$$T = 4\frac{\partial}{\partial t}, \; X_i = \frac{\partial}{\partial x_i} - 2y_i\frac{\partial}{\partial t}, \;\; Y_i = \frac{\partial}{\partial y_i} + 2x_i\frac{\partial}{\partial t}, \;\; 1 \le i \le n.$$

A function $f \in C^2(R^m)$ is strictly convex if there is a $C_0 > 0$ such that $D^2 f \ge C_0$. Now we are ready to state the uniqueness theorem.

Theorem C. *Let $\mathbf{G}$ be a Carnot group and $\Omega \subset G$ be a bounded domain. Assume that $f \in C^2(R^m, R_+)$ is strictly convex, homogeneous of degree $\alpha \ge 1$, and $f(p) > 0$ for $p \ne 0$. Then, for any $\phi \in W_{CC}^{1,\infty}(\Omega)$, the Dirichlet problem*

$$A(Xu, (D^2u)^*) := -\sum_{ij=1}^{m} f_{p_i}(Xu) f_{p_j}(Xu) X_i X_j u = 0, \quad \text{in} \;\; \Omega, \tag{1.13}$$

$$u = \phi, \quad \text{on} \; \partial\Omega.$$

has at most one viscosity solution in $C(\bar{\Omega})$*.*

Although the operator A is degenerate subelliptic, one can check that the operator $\bar{A}(x, Du, D^2u) \equiv A(Xu, (D^2u)^*)$ has x-dependence and is not degenerate elliptic (see [CIL] for its definition). Therefore, the uniqueness theorems, by Jensen [J1], Ishii [I], or Jensen -Lions-Souganidis [JLS], on viscosity solutions to 2nd order elliptic PDEs, are not applicable directly here. Our ideas are: (i) We observe that eqn.(1.13) is invariant under group multiplications: for any $a \in \mathbf{G}$, if $u \in C(\mathbf{G})$ is a viscosity solution to eqn.(1.13), then $u_a(x) = u(a \cdot x) : \mathbf{G} \to R$ is also a viscosity solution to eqn.(1.13). This enables us to extend the sup/inf convoluation construction by [JLS] to $\mathbf{G}$ to convert viscosity sub/supersolutions of eqn.(1.13) into semiconvex/concave sub/supersolutions. (ii) We modify Jensen's original arguments [J1] [J2] to prove a comparison principle between semiconvex subsolutions and semiconcave *strict* supersolutions to any 2nd order degenerate subelliptic equations, which is valid for any vector fields satisfying Hörmander's condition. (iii) We adopt Jensen's approximation scheme by p-Laplacians ([J2]) to build viscosity solutions to two auxiiary equations, with horizontal gradient constraints, having the properties that any supersolution can be converted into *strict* supersolution under small perturbations. (iv) Finally, we apply the comparison principle for the two auxiliary equations to prove the uniqueness of eqn.(1.13).

As a consequence of theorem A and C, we have

Corollary D. *Let* $\mathbf{G}$ *be a Carnot group and* $\Omega \subset \mathbf{G}$ *be a bounded domain. Assume that* $f \in C^2(R^m, R_+)$ *is strictly convex, homogeneous of degree* $\alpha \geq 1$*, and* $f(p) > 0$ *for* $p \neq 0$*. Then, for any* $\phi \in W^{1,\infty}_{CC}(\Omega)$*, there is a unique absolute minimizer* $u \in W^{1,\infty}_{CC}(\Omega)$*, with* $u|_{\partial\Omega} = \phi$*, to the functional* $F(v, \Omega) = \sup_{x \in \Omega} f(Xv)$*, and the eqn.(1.13) has a unique viscosity solution in* $C(\bar{\Omega})$*. In particular,* ϕ *has a unique AMLE in* $W^{1,\infty}_{CC}(\Omega)$ *and the subelliptic* ∞*-Laplacian eqn. (1.7) has a unique viscosity solution.*

We would like to remark that Bieske [B1] [B2] has previously proved the uniqueness of both AMLE and viscosity solution to eqn. (1.7) for Heisenberg group $\mathbf{H}^n$ and Grushin type plane. However, our methods are considerably different. Manfredi, in a forthcoming paper [M], studies some uniqueness issues for uniformly subelliptic 2nd order PDEs on Carnot groups.

The paper is written as follows. In §2, we outline the proof of theorem A. In §3, we discuss the sup/inf convolution construction on Carnot group $\mathbf{G}$. In §4, we discuss the comparison principle between semiconvex subsolutions and *strict* semiconcave supersolutions to any degenerate subelliptic equations associated to vector fields satisfying Hörmander's condition. In §5, we study two auxiliary equations to eqn. (1.13), with horizontal gradient constraints. In §6, we prove theorem C.

§2. Proof of theorem A

This section is devoted to the proof of theorem A. It contains two steps: (i) the construction of test functions by solving the Hamilton-Jacobi equation, which is motivated by [BJW] and [C]; (ii) the comparison between viscosity subsolution and classical strict supersolution of the Hamilton-Jacobi equation, which is motivated by [CIL] and [BJW].

Proof of theorem A. It suffices to prove that if u fails to be a viscosity subsolution of eqn.(1.12) at the point $x = 0 \in \Omega$ then u fails to be an absolute minimizer of $F(\cdot, \Omega)$. This assumption implies that there is an $r_0 > 0$ and $\phi \in C^2(\Omega)$ for which $B_{r_0}(0) \subset\subset \Omega$ such that

$$0 = u(0) - \phi(0) \geq u(x) - \phi(x), \ \forall x \in \Omega, \tag{2.1}$$

but

$$-\sum_{i=1}^{m} X_i(f(x, X\phi))f_{p_i}(x, X\phi)(0) = C_0 > 0. \tag{2.2}$$

Now we have

Lemma 2.1. *There exist a neighborhood V of 0 and an $\Phi \in C^2(V)$ such that*

$$\Phi(0) = \phi(0), \quad D\Phi(0) = D\phi(0), \quad D^2\Phi(0) > D^2\phi(0), \tag{2.3}$$

and

$$f(x, X\Phi(x)) = f(0, X\phi(0)) > 0, \ \forall x \in V. \tag{2.4}$$

Proof. Since $\{X_i\}_{i=1}^m \subset C^2(\Omega)$, there is $(a_{ij})_{1\leq i\leq m, 1\leq j\leq n} \in C^2(\Omega, R^{mn})$ such that

$$X_i(x) = \sum_{j=1}^{n} a_{ij}(x)\frac{\partial}{\partial x_j}, \ \forall x \in \Omega.$$

Define $\bar{f} : \Omega \times R^n \to R$ by

$$\bar{f}(x, q_1, \cdots, q_n) = f(x, \sum_{j=1}^{n} a_{1j}(x)q_j, \cdots, \sum_{j=1}^{n} a_{mj}(x)q_j), \quad \forall (x, q_1, \cdots, q_n) \in \Omega \times R^n.$$

Note that, for any $(x, q) \in \Omega \times R^n$ and $1 \leq i \leq n$, we have

$$\frac{\partial \bar{f}}{\partial q_i}(x, q) = \sum_{k=1}^{m} a_{ki}(x)\frac{\partial f}{\partial p_k}(x, \sum_{j=1}^{n} a_{1j}(x)q_j, \cdots, \sum_{j=1}^{n} a_{mj}(x)q_j).$$

Moreover, since $\phi \in C^2(\Omega)$, it is easy to see that

$$\bar{f}(x, D\phi(x)) = f(x, X_1\phi(x), \cdots, X_m\phi(x)) = f(x, X\phi(x)), \ \forall x \in \Omega. \tag{2.5}$$

Therefore, for any $x \in \Omega$, we have

$$\sum_{j=1}^{m} X_j(f(x, X\phi(x)))f_{p_j}(x, X\phi(x)) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\bar{f}(x, D\phi(x))\frac{\partial \bar{f}}{\partial q_i}(x, D\phi(x)). \tag{2.6}$$

This, combined with (2.2), implies

$$A(0, D\phi(0), D^2\phi(0)) := -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\bar{f}(x, D\phi))\frac{\partial \bar{f}}{\partial q_i}(x, D\phi)(0) = C_0 > 0. \tag{2.7}$$

Now we can apply exactly the step one of Crandall's argument ([C], page 275-276) to conclude that there are a neighborhood V of 0 and an $\Phi \in C^2(V)$ such that

$$\Phi(0) = \phi(0), \quad D\Phi(0) = D\phi(0), \quad D^2\Phi(0) > D^2\phi(0),$$

and

$$\bar{f}(x, D\Phi(x)) = \bar{f}(0, D\phi(0)), \quad \forall x \in V. \tag{2.8}$$

(2.8), combined with (2.5), gives (2.4).

To see $f(0, X\phi(0)) > 0$, we observe that (2.2) implies

$$f_p(0, X\phi(0)) := (\frac{\partial f}{\partial p_1}(0, X\phi(0)), \cdots, \frac{\partial f}{\partial p_m}(0, X\phi(0))) \neq 0.$$

This, combined with the fact that $f_p(0,0) = 0$, implies $X\phi(0) \neq 0$. Note that the homogenity of f implies that $f(0,0) = 0$. Therefore, $f(0, X\phi(0)) > 0$. This finishes the proof of Lemma 2.1. ■

It follows from Lemma 2.1 that there exists an open neighborhood $V_1 \subset V$ of 0 such that $\Phi(x) > \phi(x) \geq u(x)$ for any $0 \neq x \in V_1$. Since $\Phi(0) = \phi(0) = u(0)$. Therefore, for any small $\epsilon > 0$, there exists another neighborhood $V_\epsilon \subset V_1$ of 0 such that

$$\Phi(x) - \epsilon < u(x), \quad \forall x \in V_\epsilon; \quad \Phi(x) - \epsilon = u(x), \quad \forall x \in \partial V_\epsilon. \tag{2.9}$$

It follows from the absolute minimality of u to $F(\cdot,\ \Omega)$ that

$$F(u, V_\epsilon) \leq F(\Phi - \epsilon, V_\epsilon) = \sup_{x \in V_\epsilon} f(x, X\Phi(x)) = f(0, X\phi(0)). \tag{2.10}$$

Now we want to show that u is a viscosity subsolution of the Hamilton-Jacobi equation (2.8) on V_ϵ. More precisely, we have

Lemma 2.2. *Under the same notations as above.* $u \in W^{1,\infty}_{CC}(V_\epsilon)$ *is a viscosity subsolution to the Hamilton-Jacobi equation*

$$f(x, Xu(x)) - f(0, X\phi(0)) = 0, \quad in \;\; V_\epsilon. \tag{2.11}$$

Proof. For any subdomain $U \subset\subset V_\epsilon$ and $0 < \delta < \operatorname{dist}(U, \partial V_\epsilon)$, here dist denotes the euclidean distance. Let $g_\delta : U \to R$ be the usual δ-mollifier of g for any function g on V_ϵ. Since $u \in W^{1,\infty}_{CC}(V_\epsilon)$, u_δ converges uniformly to u on U as $\delta \to 0$. Since f is quasiconvex in its 2nd variable by (f1), it follows from the Jensen inequality for quasiconvex functions (cf. [BJW] theorem 1.1) that for any $x \in U$

$$f(x, (Xu)_\delta(x)) \le F(u, V_\epsilon) \le f(0, X\phi(0)).$$

Hence

$$\sup_{x\in U} f(x, (Xu)_\delta(x)) \le f(0, X\phi(0)). \tag{2.12}$$

On the other hand, for any $1 \le i \le m$ and $x \in U$, we can estimate $(X_i u)_\delta(x) - X_i(u_\delta)(x)$ as follows

$$\begin{aligned}
&(X_i u)_\delta(x) - X_i(u_\delta)(x)\\
&= \int_{R^n} \eta_\delta(x-y)\Big(\sum_{j=1}^n a_{ij}(y)\frac{\partial}{\partial y_j}\Big)(u(y)-u(x))\,dy\\
&- \int_{R^n} \sum_{j=1}^n a_{ij}(x)\frac{\partial \eta_\delta(x-y)}{\partial x_j}(u(y)-u(x))\,dy\\
&= \sum_{j=1}^n \int_{R^n} \Big\{-\frac{\partial}{\partial y_j}(a_{ij}(y)\eta_\delta(x-y)) - a_{ij}(x)\frac{\partial \eta_\delta(x-y)}{\partial x_j}\Big\}(u(y)-u(x))\,dy\\
&= \sum_{j=1}^n \int_{R^n} (a_{ij}(y) - a_{ij}(x))\frac{\partial \eta_\delta(x-y)}{\partial x_j}(u(y)-u(x))\,dy\\
&+ \sum_{j=1}^n \int_{R^n} \frac{\partial a_{ij}(y)}{\partial y_j}\eta_\delta(x-y)(u(y)-u(x))\,dy.
\end{aligned}$$

Therefore we have

$$|(X_i u)_\delta(x) - X_i(u_\delta)(x)| \le C \max_{1\le j\le n} \|Da_{ij}\|_{L^\infty(\Omega)} \int_{R^n} \{|\eta_\delta(x-y)|u(y)-u(x)|$$

$$
\begin{aligned}
&+ |y-x||D\eta_\delta(x-y)||u(y)-u(x)|\} \, dy \\
&\le C\|X_i\|_{C^1(\Omega)} \sup_{\|y-x\|\le\delta} |u(y)-u(x)| \\
&\le C\|X_i\|_{C^1(\Omega)} \|u\|_{W^{1,\infty}_{\mathrm{CC}(V_\epsilon)}} \delta^{\frac{1}{r}},
\end{aligned}
$$

where $r \ge 1$ is the step of Hörmander's condition. This implies

$$
\begin{aligned}
f(x, X(u_\delta)(x)) &\le \sup_{x\in U} f(x, (Xu)_\delta(x)) + \|f_p\|_{L^\infty} \|X(u_\delta) - (Xu)_\delta\|_{L^\infty(U)}. \\
&\le f(0, X\phi(0)) + C\delta^{\frac{2}{r}}, \quad \forall x \in U. \qquad (2.13)
\end{aligned}
$$

This, combined with the compactness theorem for viscosity solutions (cf. [CIL]), yields that u is a viscosity subsolution to the eqn. (2.8) in U. Since U exhausts V_ϵ as $\delta \to 0$, we have that u is a viscosity subsolution of the eqn. (2.8) in V_ϵ. The proof of Lemma 2.2 is complete. ■

Now we continue the proof of theorem A. It follows from (f2) that

$$
f(x, (1+t)p) = (1+t)^\alpha f(x,p) = (1+g(t)) f(x,p), \; \forall t > 0, \; \forall (x,p) \in \Omega \times R^n,
$$

where $g(t) \equiv (1+t)^\alpha - 1 > 0$ for $t > 0$, for $\alpha \ge 1$. This, combined with (2.8), implies that, for any $t > 0$,

$$
f(x, X((1+t)\Phi_\epsilon)(x)) = (1+g(t)) f(0, X\phi(0)) = f(0, X\phi(0)) + \delta(t), \; \forall x \in V_\epsilon, \qquad (2.14)
$$

where $\Phi_\epsilon \equiv \Phi - \epsilon$ and $\delta(t) = g(t) f(0, X\phi(0)) > 0$. Therefore, for any $t > 0$, $(1+t)\Phi_\epsilon$ is a *strict*, *classical* supersolution of eqn. (2.8). We can now apply the comparison theorem for the Hamilton-Jacobi eqn.(2.8) (see, e.g. Crandall-Ishii-Lions [CIL]) to conclude that

$$
\sup_{V_\epsilon} (u - (1+t)\Phi_\epsilon) \le \sup_{\partial V_\epsilon} (u - (1+t)\Phi_\epsilon), \; \forall t > 0. \qquad (2.15)
$$

Taking t into zero, we have

$$
\sup_{V_\epsilon} (u - \Phi_\epsilon) \le \sup_{\partial V_\epsilon} (u - \Phi_\epsilon) = 0.
$$

This implies

$$
u(x) \le \Phi_\epsilon(x), \; \forall x \in V_\epsilon.
$$

This clearly contradicts with (2.9). Therefore the proof of theorem A is complete. ■

§3. The construction of sup/inf convolutions on **G**

This section is devoted to the construction of sup/inf convolutions on the Carnot group **G**, which is the necessary extension of Jensen-Lions-Souganidis [JLS] we need for the proof of theorem C.

Let $\Omega \subset \mathbf{G}$ be a bounded domain and $d : \mathbf{G} \times \mathbf{G} \to R_+$ be the gauge distance defined in §1. For any $\epsilon > 0$, define

$$\Omega_\epsilon = \{x \in \Omega : \inf_{y \in \mathbf{G} \setminus \Omega} d^{2r!}(x^{-1}, y^{-1}) \geq \epsilon\}.$$

Definition 3.1. For any $u \in C(\bar{\Omega})$ and $\epsilon > 0$, the sup involution, u_ϵ, of u is defined by

$$u^\epsilon(x) = \sup_{y \in \bar{\Omega}} (u(y) - \frac{1}{2\epsilon} d(x^{-1}, y^{-1})^{2r!}), \ \forall x \in \Omega. \tag{3.1}$$

Similarly, the inf involution, v_ϵ, of $v \in C(\bar{\Omega})$ is defined by

$$v_\epsilon(x) = \inf_{y \in \bar{\Omega}} (v(y) + \frac{1}{2\epsilon} d(x^{-1}, y^{-1})^{2r!}), \ \forall x \in \Omega. \tag{3.2}$$

For $x \in \mathbf{G}$, let $|x| := (\sum_{j=1}^{r} \sum_{i=1}^{m_j} x_{ij}^2)^{\frac{1}{2}}$ denote its euclidean norm. We recall

Definition 3.2. A function $u \in C(\bar{\Omega})$ is called semiconvex, if there is a constant $C > 0$ such that $u(x) + C|x|^2$ is convex; u is called semiconcave if $-u$ is semiconvex. Note that, for $u \in C^2(\Omega)$, if $D^2 u(x) \geq -C$ for $x \in \Omega$ then u is semiconvex, here $D^2 u$ denotes the (full) hessian of u.

Now we have the generalized version of [JLS].

Proposition 3.3. *For $u, v \in C(\bar{\Omega})$, denote $R_0 = 2 \max\{\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}\}$. Then, for any $\epsilon > 0$, $u^\epsilon, v_\epsilon \in W^{1,\infty}_{CC}(\Omega)$ satisfying:*

(1) u^ϵ is semiconvex and v_ϵ is semiconcave.

(2) u^ϵ is monotonically nondecreasing w.r.t. ϵ and converges uniformly to u on $\Omega_{(1+4R_0)\epsilon}$; v_ϵ is monotonically nonincreasing w.r.t. ϵ and converges uniformly to v on $\Omega_{(1+4R_0)\epsilon}$.

(3) if u (or v respectively) is a viscosity subsolution (or supersolution respectively) to a degenerate subelliptic equation:

$$B(Xu, (D^2 u)^*) = 0 \quad in \quad \Omega, \tag{3.3}$$

then u^ϵ (or v_ϵ) is a viscosity subsolution (or supersolution respectively) to eqn. (3.3) in $\Omega_{(1+4R_0)\epsilon}$.

Proof. Since the proof of v_ϵ is similar to that of u^ϵ, we only prove the conclusions for u^ϵ.
(1) Since $\Omega \subset \mathbf{G}$ is bounded, it is easy to see from the formula of d that

$$C_d(\Omega) \equiv \|D_x^2(d(x^{-1}, y^{-1})^{2r!})\|_{L^\infty(\bar{\Omega} \times \bar{\Omega})} < \infty.$$

Therefore, for any $y \in \bar{\Omega}$,

$$\tilde{u}^{\epsilon}_y(x) := u(y) - \frac{1}{2\epsilon} d(x^{-1}, y^{-1})^{2r!} + \frac{C_d(\Omega)}{2\epsilon}|x|^2, \ \forall x \in \Omega,$$

has nonnegative hessian and is convex. Since the maximum for a family of convex functions is still convex, this implies that

$$u_\epsilon(x) + \frac{C_d(\Omega)}{2\epsilon}|x|^2 = \sup_{y \in \bar{\Omega}} \tilde{u}^{\epsilon}_y(x)$$

is convex so that u_ϵ is semiconvex. It is well-known that semiconvex functions are Lipschitz with respect to the euclidean metric so that $u^\epsilon \in W^{1,\infty}_{\mathrm{CC}}(\Omega)$.

(2) It is easy to see that for any $\epsilon_1 < \epsilon_2$ $u^{\epsilon_1}(x) \le u^{\epsilon_2}(x)$ and $u(x) \le u^\epsilon(x) \le R_0$ for any $x \in \Omega$. Observe that for any $x \in \Omega$, $u^\epsilon(x) = \sup_{\bar{\Omega} \cap \{d^{2r!}(x^{-1}, y^{-1}) \le 4R_0\epsilon\}}(u(y) - \frac{1}{2\epsilon} d(x^{-1}, y^{-1})^{2r!})$. Therefore, for any $x \in \Omega_{(1+4R_0)\epsilon}$, $u^\epsilon(x)$ is attained at points $y \in \Omega$. To see $u^\epsilon \to u$ uniformly on $\Omega_{(1+4R_0)\epsilon}$, we observe that if $u^\epsilon(x)$ is attained by x_ϵ then

$$u_{\frac{\epsilon}{2}}(x) \ge u(x_\epsilon) - \frac{1}{\epsilon} d(x^{-1}, x_\epsilon^{-1})^{2r!} = u_\epsilon(x) - \frac{1}{2\epsilon} d(x^{-1}, x_\epsilon^{-1})^{2r!}.$$

Hence

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} d(x^{-1}, x_\epsilon^{-1})^{2r!} = 0.$$

This implies that $x_\epsilon \to x$ and $\lim_{\epsilon \to 0} u^\epsilon(x) = u(x)$. Moreover, since

$$|u^\epsilon(x_1) - u^\epsilon(x_2)| \le |u(x_1) - u(x_2)|, \ \forall x_1, x_2 \in \Omega,$$

the convergence is uniform on $\Omega_{(1+4R_0)\epsilon}$.

(3) For any $x_0 \in \Omega_{(1+4R_0)\epsilon}$, let $\phi \in C^2(\Omega_{(1+4R_0)\epsilon})$ be such that

$$u_\epsilon(x_0) - \phi(x_0) \ge u_\epsilon(x) - \phi(x), \quad \forall x \in \Omega_{(1+4R_0)\epsilon}.$$

It follows from the proof of (2) above that there exists a $y_0 \in \Omega$ such that

$$u_\epsilon(x_0) = u(y_0) - \frac{1}{2\epsilon} d(x_0^{-1}, y_0^{-1})^{2r!}.$$

Therefore, we have

$$u(y_0) - \frac{1}{2\epsilon} d(x_0^{-1}, y_0^{-1})^{2r!} - \phi(x_0) \ge u(y) - \frac{1}{2\epsilon} d(x^{-1}, y^{-1})^{2r!} - \phi(x), \forall x, y \in \Omega_{(1+4R_0)\epsilon}.$$

For y near y_0, since $x = x_0 \cdot y_0^{-1} \cdot y \in \Omega_{(1+4R_0)\epsilon}$, we have

$$u(y_0) - \phi(x_0 \cdot y_0^{-1} \cdot y_0) \ge u(y) - \phi(x_0 \cdot y_0^{-1} \cdot y).$$

Set $\tilde{\phi}(y) = \phi(x_0 \cdot y_0^{-1} \cdot y)$ for $y \in \Omega_{(1+4R_0)\epsilon}$ near y_0. Then $\tilde{\phi}$ touches u from above at $y = y_0$ and we have

$$B(X\tilde{\phi}, (D^2\tilde{\phi})^*)(y_0) \leq 0. \tag{3.4}$$

Now using the left-invariance of X_i, we know

$$X\tilde{\phi}(y) = (X\phi)(x_0 \cdot y_0^{-1} \cdot y), \quad (D^2(\tilde{\phi}))^*(y) = (D^2\phi)^*(x_0 \cdot y_0^{-1} \cdot y).$$

This implies

$$B(X\phi(x_0), (D^2\phi)^*(x_0)) \leq 0.$$

Hence u^ϵ is a viscosity subsolution of eqn.(3.3) on $\Omega_{(1+4R_0)\epsilon}$ and the proof of the proposition is complete. ■

§4. Comparison principle between semiconvex subsolutions and semiconcave supersolutions

In this section, we establish the comparison principle between semiconvex subsolutions and semiconcave *strict* supersolutions for any 2nd order subelliptic, possibly degenerate, PDE on the Carnot-Carathédory metric space generated by vector fields satisfying Hörmander's condition. The argument is inspired by the well-known maximum principle for semiconvex functions, due to Jensen [J1] [J2], on 2nd order elliptic PDEs. Here we assume that $\{X_i = \sum_{j=1}^n a_{ij}(x)\frac{\partial}{\partial x_j}\}_{i=1}^m$ is a set of vector fields on R^n satisfying Hörmander's condition. The main proposition of this section is

Proposition 4.1. *For a bounded domain $\Omega \subset R^n$. Suppose that $B \in C(\Omega \times \mathcal{S}^m)$ is degenerate subelliptic. Assume that $u \in C(\bar{\Omega})$ is a semiconvex subsolution to*

$$B(Xw, (D^2w)^*) = 0, \quad in \ \ \Omega, \tag{4.1}$$

and $v \in C(\bar{\Omega})$ is a semiconcave supersolution to

$$B(Xw, (D^2w)^*) - \mu = 0, \quad in \ \ \Omega, \tag{4.2}$$

for some $\mu > 0$. Then

$$\sup_\Omega (u - v) \leq \sup_{\partial\Omega} (u - v). \tag{4.3}$$

Proof. Suppose that (4.3) were false. Then

$$\sup_\Omega (u - v) > \sup_{\partial\Omega} (u - v),$$

so that $u-v$ achieves its maximum on $\bar{\Omega}$ at a $x_0 \in \Omega$. Since $u-v$ is semiconvex, it is well-known (cf. [J2] page 67) that

$$Du(x_0), Dv(x_0) \text{ both exist and are equal,}$$

$$u(x)-u(x_0)-\langle Du(x_0), x-x_0\rangle = O(|x-x_0|^2), \tag{4.4}$$

$$v(x)-v(x_0)-\langle Dv(x_0), x-x_0\rangle = O(|x-x_0|^2), \tag{4.5}$$

where $\langle\cdot,\cdot\rangle$ and $|\cdot|$ denote the Euclidean inner product and Euclidean norm. Let $R_0 = \text{dist}(x_0, \partial\Omega) > 0$ be the euclidean distance from x_0 to $\partial\Omega$ and $R_1 > 0$ be such that both (4.4) and (4.5) hold, with $|x-x_0| < R_1$. Set $R_2 = \min\{R_0, R_1\} > 0$. Then, for any $\rho > 0$, define the rescaled maps $u^\rho, v^\rho : B_{R_2\rho^{-1}} \to R$ by

$$u^\rho(x) = \frac{1}{\rho^2}(u(x_0+\rho x) - u(x_0) - \rho\langle Du(x_0), x\rangle),$$

$$v^\rho(x) = \frac{1}{\rho^2}(v(x_0+\rho x) - v(x_0) - \rho\langle Dv(x_0), x\rangle),$$

where the Euclidean addition and scalar multiplication are used. Then, it is easy to see

$$0 = (u^\rho - v^\rho)(0) \geq (u^\rho - v^\rho)(x), \quad \forall x \in B_{R_2\rho^{-1}}.$$

It follows from (4.4) and (4.5) that, for any $R > 0$, there exists an $\rho_0 = \rho_0(R) > 0$ such that (i) $\{u^\rho\}_{\{0<\rho\leq\rho_0\}}$ are uniformly bounded, uniformly semiconvex, and uniformly Lipschitz continuous in B_R; (ii) $\{v^\rho\}_{\{0<\rho\leq\rho_0\}}$ are uniformly bounded, uniformly semiconcave, and uniformly Lipschitz continuous in B_R. Therefore, by the Cauchy diagonal process, we may assume that there is $\rho_i \downarrow 0$ such that $u^{\rho_i} \to u^*$, $v^{\rho_i} \to v^*$ locally uniformly in R^n. Moreover, it is not difficult to see that u^* is locally bounded, semiconvex in R^n, v^* is locally bounded, semiconcave in R^n, and

$$0 = (u^* - v^*)(0) \geq (u^* - v^*)(x), \quad \forall x \in R^n.$$

Now we need

Claim 4.2. *u^* is a viscosity subsolution to*

$$B_1(D^2 w) = 0, \quad in \quad R^n, \tag{4.6}$$

and v^* is a viscosity supersolution to

$$B_2(D^2 w) + \mu = 0, \quad in \quad R^n, \tag{4.7}$$

where $B_1, B_2 : \mathcal{S}^m \to R$ are defined by

$$B_1(M) = B(Xu(x_0), \{\sum_{k,l=1}^{n} (a_{ik}(x_0)a_{jl}(x_0)M_{kl} + a_{ik}(x_0)\frac{\partial a_{jl}}{\partial x_k}(x_0)\frac{\partial u}{\partial x_l}(x_0))\}_{1\le i,j\le m}),$$

$$B_2(M) = B(Xv(x_0), \{\sum_{k,l=1}^{n} (a_{ik}(x_0)a_{jl}(x_0)M_{kl} + a_{ik}(x_0)\frac{\partial a_{jl}}{\partial x_k}(x_0)\frac{\partial v}{\partial x_l}(x_0)\}_{1\le i,j\le m})).$$

This claim follows from the compactness theorem (cf. [CIL]) for a family of viscosity sub/supersolutions to 2nd order PDEs. Since u^ρ is a viscosity subsolution to

$$B(Xu(x_0)+\rho X^\rho_w, \{\sum_{k,l=1}^{n} a^\rho_{ik}a^\rho_{jl}\frac{\partial^2 w}{\partial x_k \partial x_l} + a^\rho_{ik}(\frac{\partial a_{jl}}{\partial x_k})^\rho(\frac{\partial u}{\partial x_l}(x_0)+\rho\frac{\partial w}{\partial x_l})\}_{1\le i,j\le m}) = 0, \quad (4.8)$$

and v^ρ is a viscosity supersolution to

$$B(Xv(x_0)+\rho X^\rho_w, \{\sum_{k,l=1}^{n} a^\rho_{ik}a^\rho_{jl}\frac{\partial^2 w}{\partial x_k \partial x_l} + a^\rho_{ik}(\frac{\partial a_{jl}}{\partial x_k})^\rho(\frac{\partial v}{\partial x_l}(x_0)+\rho\frac{\partial w}{\partial x_l})\}_{1\le i,j\le m}) = 0, \quad (4.9)$$

where $X^\rho = (X^\rho_1, \cdots, X^\rho_m)$, $X^\rho_i(x) = X_i(x_0+\rho x)$, $a^\rho_{ik}(x) = a_{ik}(x_0+\rho x)$, and $(\frac{\partial a_{jl}}{\partial x_k})^\rho(x) = \frac{\partial a_{jl}}{\partial x_k}(x_0+\rho x)$.

To see (4.8). Let $x_1 \in B_{R_2\rho^{-1}}$ and $\phi \in C^2(B_{R_2\rho^{-1}})$ be such that

$$0 = u^\rho(x_1) - \phi(x_1) \ge u^\rho(x) - \phi(x), \ \forall x \in B_{R_2\rho^{-1}}.$$

It is straightforward to see

$$\phi^\rho(x) \equiv u(x_0) + \langle Du(x_0), x - x_0\rangle + \rho^2\phi(\frac{x-x_0}{\rho})$$

satisfies

$$0 = u(x_0+\rho x_1) - \phi^\rho(x_0+\rho x_1) \ge u(x) - \phi^\rho(x), \ \forall x \in B_{R_0}(x_0).$$

This, combined with the fact that u is a viscosity subsolution to (4.6), implies

$$B(X\phi^\rho, (D^2\phi^\rho)^*)(x_0+\rho x_1) \ge 0. \quad (4.10)$$

Direct calculations yield

$$\frac{\partial \phi^\rho}{\partial x_k}(x) = \frac{\partial u}{\partial x_k}(x_0) + \rho\frac{\partial \phi}{\partial x_k}(\frac{x-x_0}{\rho}),$$

$$\frac{\partial^2 \phi^\rho}{\partial x_k \partial x_l}(x) = \frac{\partial^2 \phi}{\partial x_k \partial x_l}(\frac{x - x_0}{\rho}).$$

Hence (4.10) implies (4.8). It is clear that, after taking $\rho_i \to 0$, (4.8)-(4.9) imply (4.6)-(4.7). This proves claim 4.2.

Since $u^* - v^*$ is semiconvex and achieves its maximum at $x = 0$, we can apply Jensen's maximum principle for semiconvex functions (cf. [J1] [J2]) to conclude that there exists $x_* \in R^n$ such that $D^2 u^*(x_*), D^2 v^*(x_*)$ both exist and $D^2(u^* - v^*)(x_*) \leq 0$. Let $M_1, M_2 : \mathcal{S}^m \to R$ be given by

$$M_1^{ij} = \sum_{k,l=1}^n (a_{ik}(x_0) a_{jl}(x_0) \frac{\partial^2 u^*}{\partial x_k \partial x_l}(x_*) + a_{ik}(x_0) \frac{\partial a_{jl}}{\partial x_k}(x_0) \frac{\partial u}{\partial x_l}(x_0)), 1 \leq i, j \leq m,$$

and

$$M_2^{ij} = \sum_{k,l=1}^n (a_{ik}(x_0) a_{jl}(x_0) \frac{\partial^2 v^*}{\partial x_k \partial x_l}(x_*) + a_{ik}(x_0) \frac{\partial a_{jl}}{\partial x_k}(x_0) \frac{\partial v}{\partial x_l}(x_0)), 1 \leq i, j \leq m.$$

Since $Du(x_0) = Dv(x_0)$, we have, for any $p \in R^m$,

$$\sum_{1 \leq i,j \leq m} (M_1^{ij} - M_2^{ij}) p_i p_j = \sum_{k,l=1}^n (\sum_{i=1}^m p_i a_{ik}(x_0))(\sum_{j=1}^m p_j a_{jl}(x_0)) \frac{\partial^2 (u-v)^*}{\partial x_k \partial x_l}(x_*) \leq 0.$$

Hence $M_1 \leq M_2$. This, combined with the subellipticity of B and $Xu(x_0) = Xv(x_0)$, implies

$$B(Xu(x_0), M_1) - B(Xv(x_0), M_2) \geq 0. \tag{4.11}$$

On the other hand

$$B_1(D^2 u^*(x_*)) - B_2(D^2 v^*(x_*)) = B(Xu(x_0), M_1) - B(Xv(x_0), M_2) \leq -\mu < 0. \tag{4.12}$$

This contradicts with (4.11) and the proof of proposition 4.3 is complete. ■

§5. Auxiliary equations with horizontal gradient constraints

Due to the degenerancy of eqn.(1.13), we can't establish a comparison principle for solutions to eqn.(1.13) directly. To get around the issue, we follow Jensen's approximation scheme ([J2]) to construct two auxiliary equations with horizontal gradient constraints, to which supersolutions can be deformed into *strict* supersolutions under small perturbations. This section is valid for Carnot-Carathédory metric spaces associated with vector fields satisfying Hörmander's condition. In this section, we assume that $\{X_i\}_{i=1}^m$ is a set of vector fields on R^n satisfying Hörmander's condition. First, we have

Lemma 5.1. *Suppose that $f \in C^2(R^m, R_+)$ is homogeneous of degree $\alpha \geq 1$. Let $v \in C(\bar{\Omega})$ be a viscosity supersolution to*

$$\min\{f(Xw) - \epsilon, \quad -\sum_{ij=1}^{m} f_{p_i}(Xw) f_{p_j}(Xw) X_i X_j w\} = 0, \quad in\ \Omega, \tag{5.1}$$

where $\epsilon > 0$. Then, for any $\delta > 0$, there exist an $\mu = \mu(\alpha, \epsilon, \delta) > 0$ and $v_\delta \in C(\bar{\Omega})$, with $\|v_\delta - v\|_{L^\infty(\Omega)} \leq \delta$, such that v_δ is a viscosity supersolution of

$$\min\{f(Xw) - \epsilon, \quad -\sum_{ij=1}^{m} f_{p_i}(Xw) f_{p_j}(Xw) X_i X_j w\} - \mu = 0, \quad in \quad \Omega. \tag{5.2}$$

Proof. It is similar to that by Jensen [J2] (see also Juutinen [J] and Bieske [B1,2]). We sketch it here. We look for $v_\delta = g_\delta(v)$, where $g_\delta \in C^\infty(R)$ is monotonically increasing such that $g_\delta^{-1} \in C^\infty(R)$. To find g_δ, let $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ touch v_δ from below at x_0. Let $\phi_\delta = g_\delta^{-1}(\phi)$. Then ϕ_δ touches v from below at x_0 and

$$\min\{f(X\phi_\delta) - \epsilon, \quad -\sum_{ij=1}^{m} f_{p_i}(X\phi_\delta) f_{p_j}(X\phi_\delta) X_i X_j \phi_\delta\}(x_0) \geq 0.$$

Since

$$X_i\phi = g_\delta'(\phi_\delta) X_i \phi_\delta, X_i X_j \phi = g_\delta'(\phi_\delta) X_i X_j \phi_\delta + g_\delta''(\phi_\delta) X_i \phi_\delta X_j \phi_\delta,$$

we have, by the α-homogenity of f,

$$f(X\phi(x_0)) = f(g_\delta'(\phi_\delta) X\phi_\delta(x_0)) = (g_\delta'(\phi_\delta))^\alpha f(X\phi_\delta(x_0)) \geq (g_\delta'(\phi_\delta))^\alpha \epsilon, \tag{5.3}$$

and

$$\begin{aligned}
& -\sum_{ij=1}^{m} f_{p_i}(X\phi) f_{p_j}(X\phi) X_i X_j \phi(x_0) \\
& = g_\delta'(\phi_\delta)^{3\alpha} (-\sum_{ij=1}^{m} f_{p_i}(X\phi_\delta) f_{p_i}(X\phi_\delta) X_i X_j \phi_\delta)(x_0) \\
& - g_\delta'(\phi_\delta)^{2\alpha} g_\delta''(\phi_\delta) (\sum_{i=1}^{m} f_{p_i}(X\phi_\delta) X_i \phi_\delta)^2 (x_0) \\
& \geq -g_\delta'(\phi_\delta)^{2\alpha} g_\delta''(\phi_\delta) \alpha^2 \epsilon^2,
\end{aligned} \tag{5.4}$$

provided that $g_\delta''(\phi_\delta) < 0$, here we have used (5.3) and the identity $\sum_{i=1}^m f_{p_i}(p) p_i = \alpha f(p)$ in the last step. Let $C_0 = 4\|v\|_{L^\infty(\Omega)} < \infty$ and define

$$g_\delta(t) = (1 + \delta)t - \frac{\delta}{4C_0} t^2$$

for $|t| \leq 2C_0$ and then extend this function suitably to a monotonically increasing function on R. Since $g'(t) \geq 1 + \frac{\delta}{2}$ and $g''(t) = -\frac{\delta}{2C_0}$ for $|t| \leq C_0$, we have

$$f(X\phi)(x_0) \geq (1 + \frac{\delta}{2})\epsilon,$$

and

$$-\sum_{ij=1}^{m} f_{p_i}(X\phi) f_{p_j}(X\phi) X_i X_j \phi(x_0) \geq \frac{\delta \alpha^2 \epsilon^2}{2C_0}.$$

Therefore, if we choose $\mu = \min\{\frac{\delta\epsilon}{2}, \frac{\delta\alpha^2\epsilon^2}{2C_0}\} > 0$, then

$$\min\{f(X\phi) - \epsilon, \ \ -\sum_{ij=1}^{m} f_{p_i}(X\phi) f_{p_j}(X\phi) X_i X_j \phi\}(x_0) \geq \mu.$$

The proof of Lemma 5.1 is complete.

Since the argument is similar, we state without proof the analogous Lemma on viscosity subsolutions.

Lemma 5.2. *Suppose that $f \in C^2(R^m, R_+)$ is of homogeneous of degree $\alpha \geq 1$. Let $u \in C(\bar{\Omega})$ be a viscosity subsolution to*

$$\max\{\epsilon - f(Xw), \ \ -\sum_{ij=1}^{m} f_{p_i}(Xw) f_{p_j}(Xw) X_i X_j w\} = 0, \quad in \ \ \Omega, \tag{5.5}$$

where $\epsilon > 0$. Then, for any $\delta > 0$, there are an $\mu = \mu(\alpha, \epsilon, \delta) > 0$ and $u_\delta \in C(\bar{\Omega})$, with $\|u_\delta - u\|_{L^\infty(\Omega)} \leq \delta$, such that u_δ is a viscosity subsolution to the equation

$$\max\{\epsilon - f(Xw), \ \ -\sum_{ij=1}^{m} f_{p_i}(Xw) f_{p_j}(Xw) X_i X_j w\} = -\mu, \quad in \ \ \Omega. \tag{5.6}$$

We end this section with existences of viscosity solutions to eqn. (1.13), (5.3), (5.5). For this, we need both convexity of f and $f(p) > 0$ for $p \neq 0$. More precisely,

Theorem 5.3. *Suppose that $f \in C^2(R^m, R_+)$ is strictly convex, homogeneous of degree $\alpha \geq 1$, and $f(p) > 0$ for $p \neq 0$. Then, for any $g \in W^{1,\infty}_{CC}(\Omega)$, we have*

(1). There exists a viscosity solution $u \in W^{1,\infty}_{CC}(\Omega)$ to eqn.(1.13) such that $u|_{\partial\Omega} = g$.

(2). There exists a viscosity solution $u_\epsilon \in W^{1,\infty}_{CC}(\Omega)$ of eqn. (5.3) such that $u_\epsilon|_{\partial\Omega} = g$.

(3). There exists a viscosity solution $v_\epsilon \in W^{1,\infty}_{CC}(\Omega)$ of the eqn. (5.5) such that $v_\epsilon|_{\partial\Omega} = g$.

(4). There exists a continuous, nondecreasing function $\beta : R_+ \to R_+$, with $\beta(0) = 0$, such that

$$\|u_\epsilon - v_\epsilon\|_{L^\infty(\Omega)} \leq \beta(\epsilon). \tag{5.7}$$

Proof. The proof is based on L^k approximation, which was first carried out by [BDM], and then by Jensen [J2] for the ∞-Laplacian case (see also [J] [B1, 2]). For completeness, we outline it here. Since (1) follows from (2) with $\epsilon = 0$ and (3) can be done exactly in the way as (2), we only sketch (2) and (4) as follows. For $1 < k < \infty$, let $u_p \in W^{1,k}_{\mathrm{CC}}(\Omega)$ be the unique minimizer to the functional

$$F_k(v) = \int_\Omega (f(Xv)^k - \epsilon^{k-1}v), \ \forall v \in W^{1,k}_{\mathrm{CC}}(\Omega), \text{ with } v|_{\partial\Omega} = g.$$

The existence of u_k can be obtained by the direct method, due to both the convexity of f and α-homogeneity of f, i.e. $f(p) = |p|^\alpha f(\frac{p}{|p|}) \geq |p|^\alpha \min_{|z|=1} f(z) \geq C|p|^\alpha$. It is easy to verify that u_k satisfies the subelliptic p-Laplacian equation

$$-\sum_{i=1}^m X_i^*(kf(Xu_k)^{k-1}f_{p_i}(Xu)) = -\epsilon^{k-1}, \quad \text{in } \ \Omega, \tag{5.8}$$

in the sense of distributions, here X_i^* is the adjoint of X_i. Let Q denote the homogeneous dimension of R^n, with respect to the vector fields $\{X_i\}_{i=1}^m$. Then it follows from the Sobolev inequality (see, e.g., [HK]) that $\{u_k\}_{k\geq Q+1}$ is bounded and equicontinuous. Therefore we may assume, after taking possible subsequences, that there exist a $u_\epsilon \in W^{1,\infty}_{\mathrm{CC}}(\Omega)$ such that

$$u_k \to u_\epsilon \ \text{ in } \ C^0(\bar\Omega) \cap_{Q+1\leq k<\infty} W^{1,k}_{\mathrm{CC}}(\Omega).$$

It is easy to see that $u_\epsilon|_{\partial\Omega} = g$. To show that u_ϵ is a viscosity solution to the eqn. (2.1)., we need

Claim 5.4. *For $k \geq Q+1$, $u_k \in C(\bar\Omega)$ is a viscosity solution to the eqn.(5.8).*

For simplicity, we only indicate that u_k is a viscosity subsolution. For, otherwise, there are $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that

$$0 = u_k(x_0) - \phi(x_0) \geq u_k(x) - \phi(x), \ \forall x \in \Omega,$$

but

$$-\sum_{i=1}^m X_i^*(kf(X\phi)^{k-1}f_{p_i}(X\phi))(x_0) + \epsilon^{k-1} = -C_0 < 0. \tag{5.9}$$

Then there exists an $\delta_0 > 0$ such that

$$-\sum_{i=1}^m X_i^*(kf(X\phi)^{k-1}f_{p_i}(X\phi))(x) + \epsilon^{k-1} \leq -\frac{C_0}{2} < 0, \ \ \forall x \in B_{\delta_0}(x_0). \tag{5.10}$$

For any small $\delta > 0$, there is a neighborhood $V_\delta(\subset B_{\delta_0}(x_0))$ of x_0 such that $\phi_\delta \equiv \phi - \delta$ satisfies

$$\phi_\delta(x) < u_k(x),\ \forall x \in V_\delta;\quad \phi_\delta(x) = u_k(x),\ \forall x \in \partial V_\delta.$$

Note that ϕ_δ also satisfies (5.10). Multiplying (5.8) by $u_k - \phi_\delta$ and integrating over V_δ, we have

$$\sum_{i=1}^{m} \int_{V_\delta} k f(Xu_k)^{k-1} f_{p_i}(Xu_k) X_i(u_k - \phi_\delta) = \epsilon^{k-1} \int_{V_\delta} (u_k - \phi_\delta). \tag{5.11}$$

On the other hand, multiplying (5.10) by $(u_k - \phi_\delta)(\leq 0)$ and integrating over V_δ, we have

$$\sum_{i=1}^{m} \int_{V_\delta} k(f(X\phi_\delta))^{k-1} f_{p_i}(X\phi_\delta) X_i(u_k - \phi_\delta) > \epsilon^{k-1} \int_{V_\delta} (u_k - V_\delta). \tag{5.12}$$

Subtracting (5.11) from (5.12), we obtain

$$0 > k \int_{V_\delta} \sum_{i=1}^{m} (f(Xu_k)^{k-1} f_{p_i}(Xu_p) - f(X\phi_\delta)^{k-1} f_{p_i}(X\phi_\delta)) X_i(u_k - \phi_\delta)) \geq 0,$$

this contradicts with the convexity of f. This finishes the proof of Claim 5.4.

Now we show that u_ϵ is a viscosity subsolution to the eqn. (5.3). Let $x \in \Omega$ and $\phi \in C^2(\Omega)$ be such that

$$0 = u_\epsilon(x) - \phi(x) \geq u_\epsilon(y) - \phi(y),\ \forall y \in \Omega.$$

We need to show

$$\min\{f(X\phi(x)) - \epsilon,\ -\sum_{ij=1}^{m} f_{p_i}(X\phi) f_{p_j}(X\phi) X_i X_j \phi(x)\} \leq 0.$$

Since this is true if $f(X\phi(x)) \leq \epsilon$, we may assume that $f(X\phi(x)) \geq (1 + 2\delta)\epsilon$ for some $\delta > 0$. We know that there exist $x_k \in \Omega$ such that $(u_k - \phi)$ achieves its maximum at x_k and $x_k \to x$. We may also assume that, for k sufficiently large,

$$f(X\phi(x_k)) \geq (1 + \delta)\epsilon.$$

It follows from claim 5.4 that

$$-\sum_{i=1}^{m} X_i^*(kf(X\phi)^{k-1} f_{p_i}(X\phi))(x_k) \geq -\epsilon^{k-1}.$$

After expansion and dividing both sides by $k(k-1)f(X\phi)^{k-2}(x_k)$, this gives

$$\sum_{ij=1}^{m} f_{p_i}(X\phi)f_{p_j}(X\phi)X_iX_j\phi(x_k) \geq -\frac{\epsilon}{k(k-1)}\{\frac{\epsilon}{f(X\phi(x_k))}\}^{k-2} + \frac{f(X\phi(x_k))}{(k-1)}\sum_{i=1}^{m} X_i^*(f_{p_i}(X\phi))(x_k).$$

This, after taking k into ∞, gives

$$\sum_{ij=1}^{m} f_{p_i}(X\phi)f_{p_j}(X\phi)X_iX_j\phi(x) \geq 0.$$

One can argue slightly differently that u_ϵ is also a viscosity supersolution to the eqn. (5.3). This finishes the proof of (2).

Since v_ϵ is a limit, as $k \to \infty$, of the minimizers v_k to

$$G_k(v) = \int_\Omega f(Xv)^k + \epsilon^{k-1}v, \ \forall v \in W^{1,k}_{\mathrm{CC}}(\Omega), \text{ with } u|_{\partial\Omega} = g,$$

v_k satisfies

$$-\sum_{i=1}^{m} X_i^*(kf(Xv_k)^{k-1}f_{p_i}(Xv_k)) = \epsilon^{k-1}, \quad \text{in} \quad \Omega. \tag{5.13}$$

Multiplying (5.11) and (5.13) by $(u_k - v_k)$, integrating over Ω, and subtracting each other, we get

$$\begin{aligned}&\int_\Omega k(f(Xu_k)^{k-1}f_{p_i}(Xu_k) - f(Xv_k)^{k-1}f_{p_i}(Xv_k))X_i(u_k - v_k)\\ &\leq 4\epsilon^{k-1}\|u_k - v_k\|_{L^1(\Omega)}.\end{aligned} \tag{5.14}$$

Now we need

Claim 5.5. *If $f \in C^2(R^m)$ is strictly convex, then for any $p, q \in R^m$*

$$(f^{k-1}(p)f_p(p) - f^{k-1}(q)f_p(q)) \cdot (p-q) \geq C|p-q|^{\alpha(k-1)+2}. \tag{5.15}$$

To see (5.15), we observe that

$$\begin{aligned}&(f^{k-1}(p)f_p(p) - f^{k-1}(q)f_p(q)) \cdot (p-q)\\ &= k^{-1}\int_0^1 \frac{d}{dt}(f^k)_p(tp+(1-t)q)\,dt \cdot (p-q)\\ &\geq \sum_{ij=1}^{m}\int_0^1 f^{k-1}(tp+(1-t)q)f_{p_ip_j}(tp+(1-t)q)\,dt(p_i-q_i)(p_j-q_j)\\ &\geq C^{k-1}\int_0^1 |tp+(1-t)q|^{\alpha(k-1)}\,dt|p-q|^2,\end{aligned} \tag{5.16}$$

where we have used the strict convexity of f:

$$\sum_{ij=1}^{m} f_{p_ip_j}(v)p_ip_j \geq C_0|p-q|^2,\ \forall p,q,v \in R^m,$$

the α-homogenity of f and the fact $f(p) > 0$ for $p \neq 0$:

$$f(v) = |v|^\alpha f(\frac{v}{|v|}) \geq \min_{|z|=1} f(z)|v|^\alpha \geq C|v|^\alpha,\ \forall v \in R^m,$$

for some $C > 0$ depending only on f. Since

$$\int_0^1 |tp + (1-t)q|^{\alpha(k-1)}\, dt \geq C|p-q|^{\alpha(k-1)},$$

(5.16) implies (5.15). Putting (5.15) into (5.14), we obtain

$$kC^{k-1}\int_\Omega |Xu_k - Xv_k|^{\alpha(k-1)+2} \leq C\epsilon^{k-1}.$$

This, combined with the Hölder inequality, implies

$$\int_\Omega |Xu_k - Xv_k| \leq k^{-\frac{1}{\alpha(k-1)+2}}(C\epsilon)^{\frac{k-1}{\alpha(k-1)+2}}|\Omega|^{\frac{\alpha(k-1)+1}{\alpha(k-1)+2}}.$$

Taking k into ∞, we have

$$\|Xu_\epsilon - Xv_\epsilon\|_{L^1(\Omega)} \leq C\epsilon^{\frac{1}{\alpha}}. \tag{5.17}$$

In view of the fact that $u_\epsilon, v_\epsilon \in W^{1,\infty}_{\mathrm{CC}}(\Omega)$, (5.17) together with the interpolation inequality and the Sobolev inequality yield that the function β must exist as asserted in (4). ■

§6. Proof of theorem C

This section is devoted to the proof of theorem C. Henceforth we assume that $\{X_i\}_{i=1}^m$ are horizontal vector fields in a bounded domain Ω of the Carnot group $\mathbf{G}$. Since we can identify $\mathbf{G}$ with R^n, $n = \dim(\mathbf{G})$, via the exponential map, the results in §4 and §5 are all applicable to $\mathbf{G}$. The idea to prove theorem C is based on the sup/inf convolution and the comparison principle for both equations (5.1) and (5.5).

Lemma 6.1. *Under the same assumptions as theorem C. For any $\epsilon > 0$, if $v \in C(\bar{\Omega})$ is a viscosity subsolution to the eqn.(5.1) and $w \in C(\bar{\Omega})$ is a viscosity supersolution to the eqn.(5.1), Then*

$$\sup_{x\in\Omega}(v-w)(x) = \sup_{x\in\partial\Omega}(v-w)(x). \tag{6.1}$$

Proof. Suppose that (6.1) were false. Then there is an $\delta_0 > 0$ such that

$$\sup_{x\in\Omega}(v-w)(x) \geq \sup_{x\in\partial\Omega}(v-w)(x) + \delta_0.$$

For any $\delta \in (0, \frac{\delta_0}{2})$, it follows from Lemma 5.1 that there are $w_\delta \in C(\bar{\Omega})$, with $\|w_\delta - w\|_{L^\infty(\Omega)} \leq \delta$, and $\mu = \mu(\delta, \epsilon, \alpha) > 0$, such that w_δ is a viscosity supersolution to the eqn.(5.2). Moreover, we have

$$\sup_{x\in\Omega}(v-w_\delta) \geq \sup_{x\in\partial\Omega}(v-w_\delta)(x) + \frac{\delta_0}{4}. \tag{6.2}$$

Now we apply proposition 3.3 to conclude that for any $\delta \in (0, \frac{\delta_0}{2})$ there are a semiconvex $v^\delta \in W^{1,\infty}_{\mathrm{CC}}(\Omega)$ and a semiconcave $\tilde{w_\delta} \in W^{1,\infty}_{\mathrm{CC}}(\Omega)$ such that

$$\lim_{\delta\to 0}\max\{\|v^\delta - v\|_{L^\infty(\Omega_{C\delta})},\ \|\tilde{w_\delta} - w_\delta\|_{L^\infty(\Omega_{C\delta})}\} = 0, \tag{6.3}$$

where $\Omega_{C\delta}$ is defined in §3. Moreover, v^δ is a viscosity subsolution to the eqn. (5.1) and $\tilde{w_\delta}$ is a viscosity supersolution to the eqn. (5.2) on $\Omega_{C\delta}$. Therefore, we can apply proposition 4.1 to conclude that

$$\sup_{\Omega_{C\delta}}(v^\delta - \tilde{w_\delta}) = \sup_{\partial\Omega_{C\delta}}(v^\delta - \tilde{w_\delta}). \tag{6.4}$$

Taking δ into zero, this yields

$$\begin{aligned}
\lim_{\delta\to 0}\sup_{\Omega_{C\delta}}(v-w) &= \lim_{\delta\to 0}\sup_{\Omega_{C\delta}}[(v-v^\delta) + (v^\delta - \tilde{w_\delta}) + (\tilde{w_\delta} - w_\delta) + (w_\delta - w)] \\
&= \lim_{\delta\to 0}\sup_{\Omega_{C\delta}}(v^\delta - \tilde{w_\delta}) \\
&= \lim_{\delta\to 0}\sup_{\partial\Omega_{C\delta}}(v^\delta - \tilde{w_\delta}) \\
&= \sup_{\partial\Omega}(v-w).
\end{aligned}$$

This yields the desired contradiction. The proof is complete. ■

Similarly, we have the comparison principle for the equation (5.5).

Lemma 6.2. *Under the same assumptions as theorem C. For any $\epsilon > 0$. Let $v \in C(\bar{\Omega})$ be a viscosity subsolution to the eqn.(5.5) and $w \in C(\bar{\Omega})$ be a viscosity supersolution to the eqn.(5.5). Then*

$$\sup_{x\in\Omega}(v-w)(x) = \sup_{x\in\partial\Omega}(v-w)(x).$$

We are now in a position to prove a maximum principle for solutions of the eqn. (1.13)

Lemma 6.3. *Under the same assumptions as theorem C. For a given* $\phi \in W^{1,\infty}_{CC}(\Omega)$, *assume that* $v \in C(\bar{\Omega})$ *is a viscosity subsolution to the eqn.(1.13) and* $w \in C(\bar{\Omega})$ *is a viscosity supersolution to the eqn.(1.13) such that* $v|_{\partial\Omega} = w|_{\partial\Omega} = \phi$. *Then*

$$v(x) \leq w(x), \ \forall x \in \Omega. \tag{6.5}$$

Proof. Let v^+ be a viscosity solution to the eqn.(5.1) and w^- be a viscosity solution to the eqn.(5.5), with $v^+|_{\partial\Omega} = w^-|_{\partial\Omega} = \phi$, obtained by theorem 5.3. Since subsolutions of eqn.(1.13) are also subsolutions to eqn.(5.1) and supersolutions to eqn.(1.13) are also supersolutions to eqn.(5.5), we can apply Lemma 6.1, 6.2 to conclude that

$$\sup_{\Omega}(v - v^+) = \sup_{\partial\Omega}(v - v^+) = 0, \ \sup_{\Omega}(w^- - w) = \sup_{\partial\Omega}(w^{-1} - w) = 0.$$

Hence we have

$$\sup_{\Omega}(v - w) \leq \sup_{\Omega}(v^+ - w^-) \leq \beta(\epsilon),$$

where β is given by theorem 5.3. Since ϵ is arbitrary, this implies

$$\sup_{\Omega}(v - w) \leq 0.$$

This finishes the proof of Lemma 6.3. ■

It is obvious that Lemma 6.3 yields the conclusion of theorem C. Therefore, the proof of theorem C is complete. ■

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